

Robin Laplacian in the Large coupling limit: Convergence and spectral asymptotic

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Abstract

We study convergence modes as well as their respective rates for the resolvent difference of Robin and Dirichlet Laplacian on bounded smooth domains in the large coupling limit. Asymptotic expansions for the resolvent, the eigenprojections and the eigenvalues of the Robin Laplacian are performed. Finally we apply our results to the case of the unit disc.

Keywords: Robin Laplacian, uniform convergence, trace class convergence, rate of convergence, asymptotic expansion.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain, with smooth boundary Γ and σ the normalized surface measure on Γ .

We consider the bilinear symmetric form defined in $L^2(\Omega) := L^2(\Omega, dx)$ by

$$D(\mathcal{E}^\beta) = H^1(\Omega), \quad \mathcal{E}^\beta(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \beta \int_{\Gamma} uv \, d\sigma, \quad \beta \geq 0. \quad (1.1)$$

Thanks to the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma, \sigma)$, the form \mathcal{E}^β is closed. Denote by H_β the selfadjoint operator associated to \mathcal{E}^β via Kato representation theorem. The operator H_β is commonly named the Laplacian with Robin boundary conditions and is characterized by

$$D(H_\beta) = \left\{ u \in H^2(\Omega), \frac{\partial u}{\partial \nu} + \beta u = 0, \text{ on } \Gamma \right\}, \quad H_\beta u = -\Delta u, \text{ on } \Omega, \quad (1.2)$$

where ν is the outer normal unit vector on Γ .

By Kato's monotone convergence theorem for sesquilinear forms (see [Kat95, Theorem

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3.13a, p.461]), the bilinear symmetric forms \mathcal{E}^β increase, as β increases to infinity, to the closed bilinear symmetric form \mathcal{E}^∞ , defined by

$$D(\mathcal{E}^\infty) = \{u \in H^1(\Omega), u = 0, \text{ on } \Gamma\}, \quad \mathcal{E}^\infty(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx. \quad (1.3)$$

Thus $D(\mathcal{E}^\infty) = H_0^1(\Omega)$ and \mathcal{E}^∞ is nothing else but the quadratic form associated to the Dirichlet Laplacian in $L^2(\Omega)$, which we denote by $-\Delta_D$. Thereby we obtain the strong convergence

$$\lim_{\beta \rightarrow \infty} (H_\beta + 1)^{-1} = (-\Delta_D + 1)^{-1}, \text{ strongly.} \quad (1.4)$$

In a wide variety of applications it turns out that it is more easy to analyze the limit than the approximating operators $(H_\beta + 1)^{-1}$. For this reason one might use the following strategy for the investigation of the operator H_β for large β : One studies the limit of the operators $(H_\beta + 1)^{-1}$ and estimates the error one makes by replacing $(H_\beta + 1)^{-1}$ by the limit. This leads to the question about how fast the operators $(H_\beta + 1)^{-1}$ converge. It is also important to find out which kind of convergence takes place. For instance convergence w.r.t. the operator norm admits much stronger conclusions about the spectral properties than strong convergence, cf., e.g., the discussion of this point in [RS80], chapter VIII.7.

In this spirit it is also interesting and practical to write down explicit asymptotic expansions for the operator $(H_\beta + 1)^{-1}$ and possibly for the eigenprojections and eigenvalues of the operator H_β for large β .

On the light of these motivations, we shall establish, in these notes, operator norm convergence as well as convergence within Schatten–von Neumann ideals of $(H_\beta + 1)^{-1}$ towards $(-\Delta_D + 1)^{-1}$ as $\beta \rightarrow \infty$. The speed of convergence for both convergence modes will be also determined. Furthermore large coupling asymptotic for spectral objects will be performed.

An aspect of novelty at this stage, among others, is that we shall establish a second order asymptotic of the eigenvalues which coefficients are explicitly computed. In its own this expansion generalizes and push forward the one given in [BC02] where the Neumann Laplacian with high conductivity inside Ω is studied.

Let us emphasize that although we shall consider regular bounded domains, our method (which basically rests on the theory elaborated in [BD05, BAB08, BAB11, BAB12, BBB14]) still works for exterior domains with smooth boundary, regarding convergence of resolvents differences.

Physically the Laplacian with Robin boundary conditions describes the interaction of a particle inside Ω with a potential of strength β concentrated on the boundary Γ . Thus for large β it describes the motion of a particle inside a set with high conductivity on the boundary (superconductivity on the boundary). We shall show among others that this phenomena is completely different from the case of having conductivity inside Ω , concerning convergence modes and convergence rates and hence spectral asymptotic.

The outline of this paper is as follows: In section 2 we give some preliminaries, whereas in section 3 we prove uniform convergence as well as convergence with respect to the Schatten–von Neumann norm of $(H_\beta + 1)^{-1} - (-\Delta_D + 1)^{-1}$. The rate of convergence,

with respect to both convergence types, is also discussed in this section. Section 4 and 5 are devoted to establish the asymptotic expansions for the resolvent, the projection and the eigenvalues of Laplacian with Robin boundary conditions for large coupling constant. In the last section we work out the case where Ω is the unit disc.

2 Preliminary

Along the paper we adopt the following notations:

- $K_1 = (-\Delta_N + 1)^{-1}$, where $-\Delta_N$ is the Neumann Laplacian on Ω .
- H_β is the selfadjoint operator in $L^2(\Omega)$ associated with \mathcal{E}^β .
- $D_\beta = K_1 - (H_\beta + 1)^{-1}$
- D_∞ is the strong limit $\lim_{\beta \rightarrow \infty} D_\beta = K_1 - (-\Delta_D + 1)^{-1}$.
- $\mathcal{E}[u] = \mathcal{E}(u, u)$, $\forall u \in H^1(\Omega)$
- $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(\Omega)}$.

We designate by J the operator *trace on the boundary* of functions from $H^1(\Omega)$:

$$J : (H^1(\Omega), \mathcal{E}_1) \rightarrow L^2(\Gamma) := L^2(\Gamma, \sigma), \quad Ju = tru. \quad (2.1)$$

As Γ is smooth, it is known that $\text{Ran} J = H^{1/2}(\Gamma)$, hence the operator JJ^* is invertible. We set

$$\check{H} := (JJ^*)^{-1}. \quad (2.2)$$

Let us also recall that the following Hardy type inequality holds true

$$\int_{\Gamma} (Ju)^2 d\sigma \leq c\mathcal{E}_1[u], \quad \forall u \in H^1(\Omega). \quad (2.3)$$

and (see [Ada75, GT01])

$$\ker(J) = H_0^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma\} \quad (2.4)$$

We shall make extensive use of the following formulae, established in [BAB11, Lemma 2.3]

$$D_\beta = J^*(1 + JJ^*)^{-1}JK_1 = (JK_1)^*\left(\frac{1}{\beta} + JJ^*\right)^{-1}(JK_1), \quad \beta > 0. \quad (2.5)$$

and [BAB11, Lemma 2.4]

$$D_\infty := \lim_{\beta \rightarrow \infty} D_\beta = (\check{H}^{1/2}JK_1)^*\check{H}^{1/2}JK_1. \quad (2.6)$$

Let $H_0^1(\Omega)^\perp$ be the \mathcal{E}_1 -orthogonal of $H_0^1(\Omega)$ and P be the \mathcal{E}_1 -orthogonal projection of $H^1(\Omega)$ into $H_0^1(\Omega)^\perp$. Then

$$J|_{H_0^1(\Omega)^\perp} : H_0^1(\Omega)^\perp \rightarrow H^{1/2}(\Gamma)$$

is an isomorphism. Its inverse operator, which we denote by \mathcal{R} is given by

$$\mathcal{R} : H^{1/2}(\Gamma) \rightarrow H^1(\Omega), \quad \psi \mapsto Pv, \quad Jv = \psi. \quad (2.7)$$

The operator \mathcal{R} is well defined. Indeed, $\mathcal{R}u$ is the unique solution in $H^1(\Omega)$ of the boundary problem

$$\begin{cases} -\Delta v + v &= 0 & \text{in } \Omega \\ v &= \psi & \text{on } \Gamma \end{cases} \quad (2.8)$$

Lemma 2.1. *The operator J is compact.*

Although the result is known we shall give an alternative proof.

Proof. Owing to the regularity of Ω and precisely to the fact that

$$\sigma(B_r(x) \cap \Gamma) \sim r^{d-1}, \quad \forall x \in \Gamma, \quad 0 < r < 1, \quad (2.9)$$

the following known (see [AF03, Theorem 5.36, p.164]) trace inequality holds true: For $d \geq 3$, $2 \leq p < \frac{2(d-1)}{d-2}$, there is a constant c such that

$$\left(\int_{\Gamma} |Ju|^p d\sigma \right)^{2/p} \leq c\mathcal{E}_1[u], \quad \forall u \in H^1(\Omega), \quad (2.10)$$

whereas the latter inequality holds true for every $2 \leq p < \infty$, for $d = 2$.

Now the compactness of J follows from [BA07, Theorem 4.1]. □

Of major importance for our method is the operator JJ^* , for which we list the relevant properties and give its precise description.

As $\text{Ran}(J)$ is dense in $L^2(\Gamma)$, the operator JJ^* is an invertible nonnegative selfadjoint operator in $L^2(\Gamma)$. Set

$$\check{H} := (JJ^*)^{-1}. \quad (2.11)$$

Then \check{H} is a nonnegative selfadjoint operator in $L^2(\Gamma)$ as well and has, by Lemma 2.1, a compact resolvent.

In general it is hard to give a clear description of the domain of \check{H} . To overcome this difficulty we shall associate to \check{H} a bilinear symmetric form, which domain is well known as well as its features.

Let us introduce the quadratic form $\check{\mathcal{E}}_1$ in $L^2(\Gamma)$, as follows:

$$D(\check{\mathcal{E}}_1) = \text{Ran}(J), \quad \check{\mathcal{E}}_1(Ju, Jv) = \mathcal{E}_1(Pu, Pv) \quad \forall u, v \in H^1(\Omega). \quad (2.12)$$

The operator $\check{\mathcal{E}}_1$ is called the trace of the Dirichlet form \mathcal{E}_1 with respect to the measure σ (see [FOT11, Chap. 6]). It is also called the Dirichlet-to-Neumann operator and was studied by many authors. For instance we refer the reader to [CF12, AM12, AtEKS14, Dan14, Auc04] and references therein.

According to [BBB14, Theorem 1.1], the quadratic form $\check{\mathcal{E}}_1$ is closed and is associated, in the sense of Kato's representation theorem, to the selfadjoint operator \check{H}^{-1} . In this special context we shall give much accurate description of the operator \check{H} .

Proposition 2.1. *1. Let $\psi \in H^{1/2}(\Gamma)$ and $u \in H^1(\Omega)$ be the unique solution of the boundary value problem*

$$\begin{cases} -\Delta u + u &= 0 & \text{in } \Omega \\ u &= \psi & \text{on } \Gamma \end{cases} \quad (2.13)$$

Then $\check{\mathcal{E}}_1[\psi] = \mathcal{E}_1[u]$. Furthermore for every $\psi \in D(\check{H})$, $\check{H}\psi = \frac{\partial u}{\partial \nu}$.

2. (Dirichlet principle). For every $\psi \in H^{1/2}(\Gamma)$, we have

$$\check{\mathcal{E}}_1[\psi] = \inf \{ \mathcal{E}_1[v] : v \in H^1(\Omega), Jv = \psi \}. \quad (2.14)$$

It follows that $\check{\mathcal{E}}_1$ is a Dirichlet form.

3. For every $u \in L^2(\Gamma)$, set $U_1^\sigma u$ the 1-potential of the signed measure $u\sigma$. Then $\check{H}^{-1}u = JU_1^\sigma u$.

4. Let G_Ω be the Neumann function of $-\Delta+1$ on Ω , i.e. the function $G : \Omega \times \Omega \rightarrow [0, \infty]$ satisfying

$$\begin{cases} -\Delta_y G(\cdot, y) + G(\cdot, y) &= \delta(\cdot, y) & \text{on } \Omega \\ \frac{\partial G(\cdot, y)}{\partial \nu} &= 0 & \text{on } \Gamma \end{cases} \quad (2.15)$$

Define the operator

$$K_\Omega^\sigma : L^2(\Gamma) \rightarrow L^2(\Gamma), \quad \psi \mapsto \int_\Gamma G_\Omega(\cdot, y) \psi(y) d\sigma(y). \quad (2.16)$$

Then $\check{H}^{-1} = K_\Omega^\sigma$.

Proof. Assertion 1. follows from the very construction of $\check{\mathcal{E}}_1$ and the use of Green's formula.

2. Clearly the left-hand-side of (2.14) is bigger than its right-hand-side. The reversed inequality follows from the existence of a minimizer together with the continuity of J .

3. Let us first observe that for every fixed $u \in H^1(\Omega)$ the signed measure $u\sigma$ has finite energy integral, i.e.

$$\left| \int_\Gamma v u d\sigma \right| \leq c(\mathcal{E}_1[v])^{1/2}, \quad \forall v \in H^1(\Omega). \quad (2.17)$$

Thus the 1-potential of $u\sigma$ is well defined and is characterized as being the unique element of $H^1(\Omega)$ such that

$$\mathcal{E}_1(U_1^\sigma u, v) = \int_{\Gamma} JuJv \, d\sigma, \quad \forall v \in H^1(\Omega). \quad (2.18)$$

Hence

$$\begin{aligned} \check{\mathcal{E}}_1(JU_1^\sigma, Jv) &= \mathcal{E}_1(U_1^\sigma u, Jv) = \int_{\Gamma} JuJv \, d\sigma \\ &= \check{\mathcal{E}}_1(\check{H}^{-1}Ju, Jv), \quad \forall v \in H^1(\Omega). \end{aligned} \quad (2.19)$$

4: Follows from the fact that

$$(-\Delta_N + 1)^{-1}u = \int_{\Omega} G_{\Omega}(\cdot, y)u(y) \, dy, \quad \forall u \in H^1(\Omega), \quad (2.20)$$

and the identity $\check{H}^{-1} = JJ^*$. \square

Henceforth we denote by e^{-tT} , $t > 0$, respectively \check{T}_t , $t > 0$, the semigroup associated to $-\Delta_N + 1$, respectively to \check{H} .

Remark 2.1. From potential theoretical results relating properties of $(T_t)_{t>0}$ to those of $(\check{T}_t)_{t>0}$, we conclude on the light of the latter Proposition that $(\check{T}_t)_{t>0}$ is Markovian and transient, however it is not conservative, i.e.,

$$0 \leq \check{T}_t 1 \neq 1, \quad \forall t > 0 \quad (2.21)$$

3 Uniform and trace class convergence

In this section we shall concentrate on various types of convergence of D_{β} to D_{∞} as well as their rates. These types are precisely convergence with respect to the operator norm and the norms of Schatten–von Neumann ideals.

Let us first quote that $\lim_{\beta \rightarrow \infty} \|D_{\beta} - D_{\infty}\| = 0$. Indeed, we already mentioned that D_{β} increases strongly to D_{∞} which is compact. Thus using [BAB11, Theorem 2.6] we get uniform convergence.

Theorem 3.1. *The operator $\check{H}JK_1$ is bounded. Consequently $(H_{\beta} + 1)^{-1}$ converges in the operator norm to $(-\Delta_D + 1)^{-1}$ with maximal rate proportional to β^{-1} . Moreover,*

$$\lim_{\beta \rightarrow \infty} \beta \|D_{\beta} - D_{\infty}\| = \|\check{H}JK_1\|^2. \quad (3.1)$$

Proof. Let $u \in H^2(\Omega)$. We claim that $Pu \in H^2(\Omega)$. Indeed, $JPu = Ju$. Therefore Pu is the unique solution in $H^1(\Omega)$ of the boundary problem:

$$\begin{cases} -\Delta v + v &= 0 & \text{in } \Omega \\ v &= u & \text{on } \Gamma \end{cases} \quad (3.2)$$

From elliptic regularity (see [GT01, Theorem 8.13]), we get that $Pu \in H^2(\Omega)$ and the claim is proved.

Let $u \in L^2(\Omega)$ and $v \in H^1(\Omega)$. Then

$$\begin{aligned}\check{\mathcal{E}}_1(JK_1u, Jv) &= \mathcal{E}_1(PK_1u, Pv) \\ &= \int_{\Omega} (\nabla PK_1u) \nabla Pv + \int_{\Omega} (PK_1u) Pv.\end{aligned}\tag{3.3}$$

As $K_1u \in H^2(\Omega)$ then $PK_1u \in H^2(\Omega)$ as well. Thus by Green's formula one obtain

$$\begin{aligned}\check{\mathcal{E}}_1(JK_1u, Jv) &= - \int_{\Omega} \Delta PK_1u Pv + \int_{\Gamma} \frac{\partial PK_1u}{\partial \nu} Pv + \int_{\Omega} PK_1u Pv \\ &= \int_{\Gamma} \frac{\partial PK_1u}{\partial \nu} v = \left(\frac{\partial PK_1u}{\partial \nu}, Jv \right)_{L^2(\Gamma)}.\end{aligned}\tag{3.4}$$

It follows that $JK_1u \in D(\check{H})$ and $\check{H}JK_1u = \frac{\partial PK_1u}{\partial \nu}$. Thus $\check{H}JK_1$ is a closed everywhere defined operator on $L^2(\Omega)$ and hence is bounded.

Finally utilizing [BAB11, Theorem 2.7], we conclude that $(H_{\beta} + 1)^{-1}$ converges uniformly to $(-\Delta_D + 1)^{-1}$ with maximal rate proportional to $\frac{1}{\beta}$ and that formula (3.1) holds true. \square

Remark 3.1. Here we observe a qualitative difference between inner superconductivity and boundary superconductivity: Whereas in our setting uniform convergence is as fast as possible, it occurs for $-\Delta + \beta 1_{\Omega_1}$, where Ω is open and $\overline{\Omega} \subset \Omega$, with a rate which is $O(\beta^{-1/2})$, according to [BC02, Agb].

For further investigations concerning convergence of resolvent differences as well as spectral asymptotic one needs strengthened regularizing properties of the operator JK_1 . To that end we establish:

Lemma 3.1. *The operator $\check{H}^{3/2}JK_1$ is bounded.*

Proof. Let $u \in L^2(\Omega)$. We have already proved that $\check{H}JK_1u = \frac{\partial PK_1u}{\partial \nu}$, which by elliptic regularity lies in the space $H^{1/2}(\Gamma) = D(\check{H}^{1/2})$.

Thus $\check{H}^{3/2}JK_1$ is a closed everywhere defined operator on $L^2(\Omega)$ and then it is bounded. \square

Before dealing with convergence within Schatten–von Neumann operator ideals, let us introduce few notations.

Let $1 \leq p < \infty$ and \mathcal{H}_i be Hilbert spaces, $i = 1, 2$. Let $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a compact operator. Then \mathcal{H}_2 has an orthonormal basis $(e_i)_{i \in I}$ such that, with $|K| := \sqrt{KK^*}$, we have

$$|K|e_i = \lambda_i e_i, \quad \forall i \in I,$$

for some suitably chosen family $(\lambda_i)_{i \in I} \subset [0, \infty)$, which is unique up to permutations. We set

$$\|K\|_{S_p} := \left(\sum_{i \in I} \lambda_i^p \right)^{1/p}.$$

The ideal $S_p(\mathcal{H}_1, \mathcal{H}_2)$, (S_p for short) denotes the set of compact operators from \mathcal{H}_1 to \mathcal{H}_2 such that $\|K\|_p < \infty$. It is called the Schatten–von Neumann class of order p .

On the light of Lemma 3.1, we are able to prove convergence with respect to the S_p norm.

Theorem 3.2. *For every $p > \frac{d-1}{2}$ we have*

$$\lim_{\beta \rightarrow \infty} \|D_\beta - D_\infty\|_{S_p} = 0. \quad (3.5)$$

In particular trace class convergence holds true for $d = 2$.

Proof. First we recall that owing to [BAB11, Corollary 2.20] that S_p -convergence holds true whenever $D_\infty \in S_p$.

Having in mind that $D_\infty = (\check{H}^{1/2}JK_1)^*\check{H}^{1/2}JK_1$, we get that it lies in S_p if and only if $\check{H}^{1/2}JK_1$ lies in S_{2p} . On the other hand from the boundedness of $\check{H}^{3/2}JK_1$, we obtain

$$\|\check{H}^{1/2}JK_1\|_{S_{2p}} \leq \|\check{H}^{-1}\|_{S_{2p}}\|\check{H}^{3/2}JK_1\|. \quad (3.6)$$

Thus we are led to prove that $\check{H}^{-1} \in S_q$ for $q > d - 1$.

To that end we shall use the trace inequality (2.10) to obtain:

a) For $d \geq 3$: from the construction of $\check{\mathcal{E}}$, the following Sobolev type inequality holds true

$$\left(\int_{\Gamma} |\psi|^{\frac{2(d-1)}{d-2}} d\sigma \right)^{\frac{d-2}{d-1}} \leq C\check{\mathcal{E}}_1[\psi], \quad \forall \psi \in H^{1/2}(\Gamma). \quad (3.7)$$

Now it is standard that Sobolev inequality leads to a lower bound for the eigenvalues $\check{\lambda}_k$ (repeated as many times as their multiplicity in an increasing way) of \check{H} :

$$\check{\lambda}_k \geq Ck^{\frac{1}{d-1}}. \quad (3.8)$$

Thus $\check{H}^{-1} \in S_q$ for every $q > d - 1$, which was to be proved for $d \geq 3$.

b) For $d = 2$, the proof is similar so we omit it. □

By the end of this section we shall discuss the rate of convergence in S_1 in two dimensions. It was proved in [BAB12, Theorem 2.3] that the maximal rate of convergence in S_1 is proportional to $1/\beta$ and that trace-class convergence holds true if and only if the operator $\check{H}JK_1$ is a Hilbert–Schmidt operator. However, according to [BAB12, Proposition 2.4] if for some $r \in (0, 1)$ the operator $\check{H}^{\frac{1+r}{2}}JK_1$ is a Hilbert–Schmidt operator then one has trace-class convergence with rate $O(1/\beta^r)$.

Proposition 3.1. *In the case $d = 2$ it holds*

$$\lim_{\beta \rightarrow \infty} \beta^r \|D_\beta - D_\infty\|_{S_1} < \infty, \quad \forall r \in (0, 1). \quad (3.9)$$

Proof. For $d = 2$ we have the lower bound

$$\check{\lambda}_k \geq Ck^{\frac{p-2}{p}}, \quad \forall p > 2.$$

Thus if for a given $r \in (0, 1)$, we choose $p > 2\frac{2-r}{1-r} > 2$, we get $(2-r)\frac{p-2}{p} > 2$. Thus $\check{H}^{\frac{r-2}{2}}$ is a Hilbert–Schmidt operator and so is $\check{H}^{\frac{1+r}{2}}JK_1$. □

Remark 3.2. We shall show in the example below that the limit exponent $r = 1$ is excluded!

4 Asymptotic expansions for the resolvents and the eigenprojections

Asymptotic expansions are theoretically and numerically interesting in the sense that they offers 'good' approximations for the studied objects. How 'good' is the approximation depends on its order and on the computation of its coefficients. In operator theory there are two types of asymptotic: uniform, i.e. the rest is small with respect the operator norm and strong asymptotic, i.e., the rest is small for every fixed element from the domain of the operator.

Although we shall give lower order asymptotic (of second order) of the spectral objects related to Robin Laplacian, we shall write explicitly the coefficients of the asymptotic and this is new to our best knowledge for such problems. In particular we shall show that the coefficients involved in the asymptotic depend only on the Neumann Laplacian and its trace, the Dirichlet Laplacian and the trace operator.

Especially, the coefficients of the expansion of the eigenvalues of the Robin Laplacian depend only on the eigenvalues of the Dirichlet Laplacian.

We start by giving an asymptotic expansion for $(H_\beta + 1)^{-1}$.

Theorem 4.1. *The following first order uniform expansion holds true:*

$$(H_\beta + 1)^{-1} = (-\Delta_D + 1)^{-1} + \frac{1}{\beta}K + \frac{1}{\beta^2}K'. \quad (4.1)$$

where, $K = (\check{H}JK_1)^*\check{H}JK_1 = \mathcal{R}\frac{\partial PK_1}{\partial \nu}$ and $\|K'\| \leq \|\check{H}^{3/2}JK_1\|^2$.

Proof. From the construction of $\check{\mathcal{E}}_1$ we derive

$$\check{\mathcal{E}}_1(JK_1u, Jv) = \mathcal{E}_1(K_1u, Pv) = (u, Pv)_{L^2(\Omega)}, \forall u \in L^2(\Omega), \forall v \in H^1(\Omega). \quad (4.2)$$

It follows that

$$(\check{H}JK_1u, Jv)_{L^2(\Gamma)} = (u, Pv)_{L^2(\Omega)} \text{ and } (\check{H}JK_1)^*Jv = Pv. \quad (4.3)$$

Then,

$$(\check{H}JK_1)^*\check{H}JK_1u = P\mathcal{R}\frac{\partial PK_1u}{\partial \nu} = \mathcal{R}\frac{\partial PK_1u}{\partial \nu}. \quad (4.4)$$

On the other hand relying on the resolvent formula (2.5)we obtain

$$D_\infty - D_\beta = (\check{H}^{1/2}JK_1)^*\check{H}^{1/2}JK_1 - (JK_1)^*\left(\frac{1}{\beta} + \check{H}^{-1}\right)^{-1}JK_1 \quad (4.5)$$

$$= (\check{H}^{1/2}JK_1)^*\check{H}^{1/2}JK_1 - (\check{H}^{1/2}JK_1)^*(1 + \frac{1}{\beta}\check{H})^{-1}\check{H}^{1/2}JK_1 \quad (4.6)$$

$$= \frac{1}{\beta}(\check{H}JK_1)^*(1 + \frac{1}{\beta}\check{H})^{-1}\check{H}JK_1 \quad (4.7)$$

$$= \frac{1}{\beta}(\check{H}JK_1)^*\check{H}JK_1 - \frac{1}{\beta^2}(\check{H}^{3/2}JK_1)^*(1 + \frac{1}{\beta}\check{H})^{-1}\check{H}^{3/2}JK_1 \quad (4.8)$$

$$(4.9)$$

To conclude, it suffices to note that, $0 \leq (1 + \frac{1}{\beta}\check{H})^{-1} \leq 1$, and the proof is finished. \square

Henceforth, $o_s(\frac{1}{\beta^2})$ (resp. $o_u(\frac{1}{\beta^2})$) denotes an operator-valued function such that $\beta^2 o_s(\frac{1}{\beta^2})f \rightarrow 0, \forall f$ (resp. $\beta^2 \|o_u(\frac{1}{\beta^2})\| \rightarrow 0$) as $\beta \rightarrow \infty$.

The latter theorem yields automatically the second order strong asymptotic expansion for large β .

Corollary 4.1. *For large β the following strong asymptotic formula holds true:*

$$\begin{aligned} (H_\beta + 1)^{-1} &= (-\Delta_D + 1)^{-1} + \frac{1}{\beta}(\check{H}JK_1)^*\check{H}JK_1 - \frac{1}{\beta^2}(\check{H}^{3/2}JK_1)^*\check{H}^{3/2}JK_1 \\ &\quad + o_s(\frac{1}{\beta^2}). \end{aligned} \quad (4.10)$$

We turn our attention now to give the expansions of the eigenprojections. To that end we need an expansion for $(H_\beta - z)$ for z in the resolvent set $\rho(H_\beta)$.

Since $\{(H_\beta + 1)^{-1}\}$ converges in norm to $(-\Delta_D + 1)^{-1}$ when $\beta \rightarrow \infty$, it follows that if $z \in \rho(-\Delta_D)$, then $z \in \rho(H_\beta)$ for β sufficiently large and $\{(H_\beta - z)^{-1}\}$ converge in norm to $(-\Delta_D - z)^{-1}$ uniformly in any compact subset of $\rho(-\Delta_D)$ as β goes to infinity. In particular the family of the resolvents $\{(H_\beta - z)^{-1}\}$ is bounded uniformly in β and z in any compact subset of $\rho(-\Delta_D)$ (for large β). Moreover, one has :

Proposition 4.1. *For large β , the resolvent $(-\Delta_\beta - z)^{-1}$ admits the second order strong asymptotic expansion uniformly in any compact subset of $\rho(-\Delta_D)$:*

$$\begin{aligned} (H_\beta - z)^{-1} &= (-\Delta_D - z)^{-1} + \frac{1}{\beta}LKL - \frac{1}{\beta^2}(LRL - (1+z)LKCLKL) \\ &\quad + o_s(\frac{1}{\beta^2}), \end{aligned} \quad (4.11)$$

where K is the operator given by Theorem 4.1 and

$$L = L(z) := (1 + (1+z)(-\Delta_D - z)^{-1}), \quad R := (\check{H}^{3/2}JK_1)^*\check{H}^{3/2}JK_1. \quad (4.12)$$

Proof. Let $z \in \rho(-\Delta_D)$, then for large β one has

$$\begin{aligned} D_\infty(z) - D_\beta(z) &= (1 + (1+z)(H_\beta - z)^{-1})(D_\infty - D_\beta) \\ &\quad \cdot (1 + (1+z)(-\Delta_D - z)^{-1}) \end{aligned} \quad (4.13)$$

By formula (4.10), it follows that:

$$u - \lim_{\beta \rightarrow \infty} \beta (D_\infty(z) - D_\beta(z)) = LKL, \quad (4.14)$$

uniformly in any compact subset of $\rho(-\Delta_D)$.

Thus, one writes,

$$(H_\beta - z)^{-1} = (-\Delta_D - z)^{-1} + \frac{1}{\beta}LKL + o_u(\frac{1}{\beta}) \quad (4.15)$$

Then, if we substitute $(D_\infty - D_\beta)$ and $(-\Delta_\beta - z)^{-1}$ by the corresponding terms given by the formulae (4.10) and (4.15) respectively, in the equation (4.13) we obtain the desired result. \square

The operators H_β converge to $-\Delta_D$ in the norm resolvent sense, furthermore these operators are selfadjoint, nonnegative with compact resolvents, then the eigenvalues of (H_β) converge to ones of $-\Delta_D$.

Let λ_∞ be an eigenvalue of $-\Delta_D$, since the operator $-\Delta_D$ has compact resolvents, then there exists $\epsilon > 0$ such that: $\text{spec}(-\Delta_D) \cap B(\lambda_\infty, \epsilon) = \{\lambda_\infty\}$, where $B(\lambda_\infty, \epsilon) = \{z \in \mathbb{C}, |z - \lambda_\infty| \leq \epsilon\}$.

In the following we denote,

- $E_\infty = \ker(-\Delta_D - \lambda_\infty)$, the eigenspace of λ_∞ , and P_∞ the spectral projection onto E_∞ . It is known that

$$P_\infty = -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} (-\Delta_D - z)^{-1} dz \quad (4.16)$$

where, $C(\lambda_\infty, \epsilon)$ is the circle of center λ_∞ and of radius ϵ .

- E_β is the direct sum of the eigenspaces associated to the eigenvalues of (H_β) contained in $B(\lambda_\infty, \epsilon)$, and P_β is the spectral projection onto E_β given by:

$$P_\beta = -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} (H_\beta - z)^{-1} dz$$

Proposition 4.2. *The spectral projection P_β admits a uniform asymptotic expansion of the form,*

$$P_\beta = P_\infty + \frac{1}{\beta} Q - \frac{1}{\beta^2} Q_1 + o_s\left(\frac{1}{\beta^2}\right) \quad (4.17)$$

Moreover, $P_\infty Q P_\infty = 0$.

Proof. Setting

$$Q = -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} LKL dz, \quad Q_1 = -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} (LRL - (1+z)LKLKL) dz, \quad (4.18)$$

then the first identity is immediate by integrating the formula (4.11) along the circle C . For the second identity, since $\lim_{\beta \rightarrow \infty} \|P_\beta - P_\infty\| = 0$, we obtain

$$P_\infty Q P_\infty = \lim_{\beta \rightarrow \infty} \beta P_\infty (P_\beta - P_\infty) P_\beta = 0. \quad (4.19)$$

□

5 Asymptotic expansion for the eigenvalues

Next we shall improve the asymptotic expansion of eigenvalues developed in [BC02, Theorem 1.2] and extend it to our context which deals with singular perturbations. The novelty at this stage is that we give a second order asymptotic expansion which coefficients are given by the eigenvalues of a matrix depending only on the Dirichlet Laplacian.

To that end we need some intermediate results.

Proposition 5.1. *The following formulae hold true:*

1.

$$PK_1 = (-\Delta_N + 1)^{-1} - (-\Delta_D + 1)^{-1} = D_\infty. \quad (5.1)$$

In particular,

$$\frac{\partial PK_1}{\partial \nu} = -\frac{\partial(-\Delta_D + 1)^{-1}}{\partial \nu}. \quad (5.2)$$

2. $P_\infty LKLP_\infty = \frac{1}{(\lambda_\infty - z)^2} MP_\infty$, where M is the matrix with entries

$$\left(\int_\Gamma \frac{\partial f_i}{\partial \nu} \frac{\partial f_j}{\partial \nu} d\sigma \right)_{1 \leq i, j \leq m} \quad (5.3)$$

in an orthonormal basis (f_1, \dots, f_m) of E_∞ .

Proof. For every $\forall u \in L^2(\Omega)$, $PK_1 u$ is the unique solution of the boundary value problem

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega \\ v = K_1 u & \text{in } \Gamma \end{cases} \quad (5.4)$$

Let v_0 be the unique solution of the equation $-\Delta v + v = -u$, in $H^2(\Omega) \cap H_0^1(\Omega)$, that is $v_0 = (-\Delta_D + 1)^{-1}(-u)$. Then, $PK_1 u$ is given by:

$$PK_1 u = v_0 + K_1 u = (-\Delta_N + 1)^{-1} u - (-\Delta_D + 1)^{-1} u, \quad (5.5)$$

yielding the first assertion.

Let f_i, f_j be eigenfunctions associated to the eigenvalue λ_∞ of $-\Delta_D$. Since $L(z)f_i = (\frac{\lambda_\infty + 1}{\lambda_\infty - z})f_i$, straightforward computations yields

$$\begin{aligned} (P_\infty LKLP_\infty f_i, f_j) &= (KL(z)f_i, L(z)f_j)_{L^2(\Omega)} \\ &= \left(\frac{\lambda_\infty + 1}{\lambda_\infty - z} \right)^2 (\check{H}JK_1 f_i, \check{H}JK_1 f_j)_{L^2(\Gamma)} \\ &= \left(\frac{\lambda_\infty + 1}{\lambda_\infty - z} \right)^2 \left(\frac{\partial(-\Delta_D + 1)^{-1} f_i}{\partial \nu}, \frac{\partial(-\Delta_D + 1)^{-1} f_j}{\partial \nu} \right)_{L^2(\Gamma)} \\ &= \frac{1}{(\lambda_\infty - z)^2} \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_j}{\partial \nu} \right)_{L^2(\Gamma)}, \end{aligned} \quad (5.6)$$

and the proof is done. \square

Proposition 5.2. *Let (f_k) be an orthonormal basis of Dirichlet eigenfunctions,*

$$-\Delta_D f_k = \lambda_k f_k,$$

and Q be the operator given by Proposition 4.2.

For f_i and f_j in E_∞ we set,

$$a_{i,j,k} := \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right)_{L^2(\Gamma)} \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right)_{L^2(\Gamma)} \quad (5.7)$$

Then we obtain,

1. $(P_\infty LRLP_\infty f_i, f_j) = \frac{1}{(\lambda_\infty - z)^2} (\frac{\partial}{\partial \nu} \mathcal{R}(\frac{\partial f_i}{\partial \nu}), \frac{\partial f_j}{\partial \nu})_{L^2(\Gamma)}$
2. $(P_\infty LKLP_\infty f_i, f_j) = \frac{1}{(z - \lambda_\infty)^2} \sum_k \frac{1}{(\lambda_k - z)(1 + \lambda_k)} a_{i,j,k}$
3. $(P_\infty LKLQP_\infty f_i, f_j) = \sum_{f_k \in E_\infty^\perp} \frac{1}{(\lambda_\infty - z)(\lambda_k - z)(\lambda_k - \lambda_\infty)} a_{i,j,k}$

Proof. 1.

$$\begin{aligned}
(P_\infty LRLP_\infty f_i, f_j) &= (RL(z)f_i, L(z)f_j) = (\frac{1 + \lambda_\infty}{\lambda_\infty - z})^2 (Rf_i, f_j) \\
&= (\frac{1 + \lambda_\infty}{\lambda_\infty - z})^2 (\check{H}^{3/2} JK_1 f_i, \check{H}^{3/2} JK_1 f_j)_{L^2(\Gamma)} \\
&= \frac{1}{(\lambda_\infty - z)^2} (\check{H}^{1/2} \frac{\partial f_i}{\partial \nu}, \check{H}^{1/2} \frac{\partial f_j}{\partial \nu})_{L^2(\Gamma)} \\
&= \frac{1}{(\lambda_\infty - z)^2} (\check{H} \frac{\partial f_i}{\partial \nu}, \frac{\partial f_j}{\partial \nu})_{L^2(\Gamma)} \\
&= \frac{1}{(\lambda_\infty - z)^2} (\frac{\partial}{\partial \nu} \mathcal{R}(\frac{\partial f_i}{\partial \nu}), \frac{\partial f_j}{\partial \nu})_{L^2(\Gamma)}.
\end{aligned} \tag{5.8}$$

In the last step we used the fact that for $\varphi \in D(\check{H})$ we have $\check{H}\varphi = \frac{\partial u}{\partial \nu}$ where $u = \mathcal{R}\varphi$. Indeed, by the definition of \mathcal{R} , $\mathcal{R}\varphi$ solves

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \Gamma \end{cases}$$

2. Making use of Proposition 5.1, an elementary computation yields

$$(Kf_i, f_k) = (\check{H} JK_1 f_i, \check{H} JK_1 f_k) = (1 + \lambda_i)^{-1} (1 + \lambda_k)^{-1} (\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}). \tag{5.9}$$

Thus

$$Kf_i = \sum_k (1 + \lambda_i)^{-1} (1 + \lambda_k)^{-1} (\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}) f_k \tag{5.10}$$

and

$$L(z)Kf_i = \sum_k (1 + \lambda_i)^{-1} (\lambda_k - z)^{-1} (\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}) f_k. \tag{5.11}$$

Set $B = P_\infty H(z)KH(z)KH(z)P_\infty$. Then

$$\begin{aligned}
(Bf_i, f_j) &= (\frac{1 + \lambda_\infty}{z - \lambda_\infty})^2 (KLKf_i, f_j) = (\frac{1 + \lambda_\infty}{z - \lambda_\infty})^2 (LKf_i, Kf_j) \\
&= \frac{1}{(z - \lambda_\infty)^2} \sum_k \frac{1}{(\lambda_k - z)(1 + \lambda_k)} (\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}) (\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu}).
\end{aligned} \tag{5.12}$$

3. Finally setting, $A(z, s) = P_\infty L(z)KL(z)L(s)KL(s)P_\infty$, we obtain

$$\begin{aligned} (Af_i, f_j) &= \frac{(1 + \lambda_\infty)^2}{(\lambda_\infty - z)(\lambda_\infty - s)} (L(s)Kf_i, L(z)Kf_j) \\ &= \frac{1}{(\lambda_\infty - z)(\lambda_\infty - s)} \sum_k \frac{1}{(\lambda_k - z)(\lambda_k - s)} \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right) \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right). \end{aligned} \quad (5.13)$$

Regarding the definition of Q , we achieve

$$\begin{aligned} (P_\infty L(z)KL(z)QP_\infty f_i, f_j) &= -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} (A(z, s)f_i, f_j) d\sigma(s) \\ &= \sum_{f_k \in E_\infty^\perp} \frac{(\lambda_k - \lambda_\infty)^{-1}}{(\lambda_\infty - z)(\lambda_k - z)} \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right) \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu} \right) \end{aligned} \quad (5.14)$$

□

Now we are in position to establish the asymptotic of the eigenvalue of the Robin Laplacian.

Theorem 5.1. *Let λ_∞ be an eigenvalue of $-\Delta_D$ with multiplicity m and eigenspace E_∞ . Then for sufficiently large β , the operator H_β has exactly m eigenvalues counted according to their multiplicities in $B(\lambda_\infty, \epsilon)$. These eigenvalues admit the asymptotic expansion*

$$\lambda_{i,j,\beta} = \lambda_\infty - \frac{1}{\beta} \alpha_i + \frac{1}{\beta^2} \mu_{i,j} + o\left(\frac{1}{\beta^2}\right), \quad (5.15)$$

where (α_i) are the repeated eigenvalues of the matrix

$$M := \left(\int_\Gamma \frac{\partial f_i}{\partial \nu} \overline{\frac{\partial f_j}{\partial \nu}} \right)_{1 \leq i,j \leq m}$$

in an orthonormal basis (f_1, \dots, f_m) of the eigenspace E_∞ .

Moreover, setting P_i the eigenprojection associated to the eigenvalue α_i and N the matrix given by

$$\begin{aligned} N &:= \left(\left(\frac{\partial}{\partial \nu} \mathcal{R} \left(\frac{\partial f_i}{\partial \nu} \right), \frac{\partial f_j}{\partial \nu} \right)_{L^2(\Gamma)} + \frac{1}{1 + \lambda_\infty} \sum_{f_k \in E_\infty} a_{i,j,k} \right. \\ &\quad \left. + \sum_{f_k \in E_\infty^\perp} \frac{(1 + \lambda_\infty)}{(1 + \lambda_k)(\lambda_\infty - \lambda_k)} a_{i,j,k} \right)_{1 \leq i,j \leq m}, \end{aligned} \quad (5.16)$$

then, $\mu_{i,j}$, $1 \leq j \leq \dim P_i$ are the repeated eigenvalues of $P_i N P_i$ in the subspace $P_i E_\infty$.

Proof. Following Kato's method (see [Kat95]), we introduce the

$$A_\beta := 1 - P_\infty + P_\beta P_\infty = 1 - (P_\infty - P_\beta)P_\infty.$$

For large β the operator A_β is invertible and maps E_∞ onto E_β since $\|P_\beta - P_\infty\|$ is small, and leaves the orthogonal of E_∞ invariant.

Using Proposition 4.2, we obtain the asymptotic expansion for A_β :

$$A_\beta = 1 + \frac{1}{\beta} QP_\infty - \frac{1}{\beta^2} Q_1P_\infty + o_s\left(\frac{1}{\beta^2}\right) \quad (5.17)$$

Since, $P_\infty QP_\infty = 0$ it follows that,

$$\begin{aligned} A_\beta^{-1} &= 1 - \left(\frac{1}{\beta} QP_\infty - \frac{1}{\beta^2} Q_1P_\infty\right) + \left(\frac{1}{\beta} QP_\infty - \frac{1}{\beta^2} Q_1P_\infty\right)^2 + o_s\left(\frac{1}{\beta^2}\right) \\ &= 1 - \frac{1}{\beta} QP_\infty + \frac{1}{\beta^2} Q_1P_\infty + o_s\left(\frac{1}{\beta^2}\right) \end{aligned} \quad (5.18)$$

Now we define the operator B_β as

$$B_\beta := P_\infty A_\beta^{-1} (-\Delta_\beta) A_\beta P_\infty.$$

Obviously B_β belongs to is bounded and has finite rank. Furthermore, the repeated eigenvalues of $(-\Delta_\beta)$ considered in the m -dimensional subspace E_β are equal to the eigenvalues of $H_\beta P_\beta$ in E_β and therefore also to those of $A_\beta^{-1} H_\beta A_\beta$ which is similar to $H_\beta P_\beta$ in E_β . Thus, taking into account that $P_\infty QP_\infty = 0$ and that P_∞ commutes with $(-\Delta_D - z)^{-1}$, we obtain the asymptotic expansion:

$$\begin{aligned} P_\infty A_\beta^{-1} (H_\beta - z)^{-1} A_\beta P_\infty &= (P_\infty + \frac{1}{\beta^2} P_\infty Q_1 P_\infty + o_s(\frac{1}{\beta^2})) \\ &\cdot ((-\Delta_D - z)^{-1} + \frac{1}{\beta} LKL - \frac{1}{\beta^2} (LRL - (1+z)LKLKL)) \\ &\cdot (1 + \frac{1}{\beta} QP_\infty - \frac{1}{\beta^2} Q_1P_\infty + o_s(\frac{1}{\beta^2})) P_\infty \\ &= P_\infty (-\Delta_D - z)^{-1} P_\infty + \frac{1}{\beta} P_\infty LKLP_\infty \\ &- \frac{1}{\beta^2} (P_\infty LRLP_\infty - (1+z)P_\infty LK LKLP_\infty) \\ &+ \frac{1}{\beta^2} P_\infty Q_1 (-\Delta_D - z)^{-1} P_\infty - \frac{1}{\beta^2} P_\infty (-\Delta_D - z)^{-1} Q_1 P_\infty \\ &+ \frac{1}{\beta^2} P_\infty LK LQP_\infty + o_u(\frac{1}{\beta^2}) \\ &= (\lambda_\infty - z)^{-1} P_\infty + \frac{1}{\beta} (\lambda_\infty - z)^{-2} MP_\infty - \frac{1}{\beta^2} P_\infty LRLP_\infty \\ &+ \frac{1}{\beta^2} ((1+z)P_\infty LK LKLP_\infty + P_\infty LK LQP_\infty) \\ &+ o_u(\frac{1}{\beta^2}). \end{aligned} \quad (5.19)$$

Here we have used the fact that $o_s(\frac{1}{\beta^2})P_\infty = o_u(\frac{1}{\beta^2})$ because P_∞ has finite rank.

Since

$$(-\Delta_\beta)P_\beta = -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} z (-\Delta_\beta - z)^{-1} dz, \quad (5.20)$$

integration of (5.19) along the circle $C(\lambda_\infty, \epsilon)$ after multiplication by $(-z/2i\pi)$ and by an elementary calculation of residues at the singularity λ_∞ we obtain,

$$B_\beta = P_\infty A_\beta^{-1}(-\Delta_\beta) P_\beta A_\beta P_\infty = \lambda_\infty P_\infty - \frac{1}{\beta} M P_\infty + \frac{1}{\beta^2} N P_\infty + o_u\left(\frac{1}{\beta^2}\right). \quad (5.21)$$

Theorem 5.1 is then a consequence of the well known results on finite dimensional spaces (see [Kat95]). □

6 Example: The case of the unit disc in \mathbb{R}^2

Let D be the unit disc and C be its boundary (the unit circle). First, we study the solutions of the eigenvalue problem $-\Delta f = \lambda f$ with either Dirichlet or Neumann boundary conditions.

By separating variables it turns out that the solutions of the equation $-\Delta f = \lambda f$ are given by

$$J_n(\sqrt{\lambda}r)e^{\pm in\theta}, \quad n \in \mathbb{N}, \quad (6.1)$$

where the J_n 's are Bessel functions of the first kind.

If $J_n(\sqrt{\lambda}) = 0$, then λ is an eigenvalue of the Dirichlet Laplacian on D with eigenfunctions $J_n(\sqrt{\lambda}r)e^{\pm in\theta}$. As every J_n has infinitely many positive solutions, we shall order them as follows $0 < k_{n,1} < k_{n,2} < \dots < k_{n,m} < \dots$ $n \in \mathbb{N}$.

Therefore the eigenvalues of the Dirichlet Laplacian on the unit disc are given by

$$\lambda_{n,m} = k_{n,m}^2, \quad n \in \mathbb{N}, \quad m \geq 1. \quad (6.2)$$

with associated eigenfunctions

$$\varphi_{n,m}^\pm(r, \theta) = J_n(k_{n,m}r)e^{\pm in\theta}, \quad n \in \mathbb{N}, \quad m \geq 1. \quad (6.3)$$

The Neumann eigenvalues are characterized by the equation $\sqrt{\lambda}J'_n(\sqrt{\lambda}) = 0$, $\lambda \geq 0$. As before we order the zeros of each J'_n in an increasing order

$$0 < k'_{n,1} < k'_{n,2} < \dots < k'_{n,m} < \dots, \quad n \geq 1 \quad (6.4)$$

$$0 = k'_{0,1} < k'_{0,2} < \dots < k'_{0,m} < \dots \quad (6.5)$$

Thus the eigenvalues of the Neumann Laplacian on the unit disc are given by

$$\mu_{n,m} = k_{n,m}'^2, \quad n \in \mathbb{N}, \quad m \geq 1, \quad (6.6)$$

with associated eigenfunctions,

$$\psi_{n,m}^\pm(r, \theta) = J_n(k'_{n,m}r)e^{\pm in\theta}, \quad n \in \mathbb{N}, \quad m \geq 1 \quad (6.7)$$

By using the formula,

$$\int_0^1 J_n^2(cr)r \, dr = \frac{1}{2}J_n'^2(c) + \frac{1}{2}\left(1 - \frac{n^2}{c^2}\right)J_n^2(c), \quad (6.8)$$

the normalized Neumann eigenfunctions associated to the eigenvalue $\mu_{n,m} = k_{n,m}'^2$ are given by,

$$\Psi_{n,m}^{\pm}(r, \theta) = \pi^{-1/2} \left(1 - \frac{n^2}{k_{n,m}'^2}\right)^{-1/2} \frac{J_n(k_{n,m}' r)}{J_n(k_{n,m}')} e^{\pm i n \theta}, \quad n \in \mathbb{N}, \quad m \geq 1 \quad (6.9)$$

From now on the notation $\sum_n \cdots (g_n^{\pm}, \cdot) f_n^{\pm}$ means

$$\sum_n \cdots (g_n^{+}, \cdot) f_n^{+} + \sum_n \cdots (g_n^{-}, \cdot) f_n^{-}.$$

Thus, by the spectral calculus we obtain

$$K_1 = (-\Delta_N + 1)^{-1} = \sum_{n,m} (1 + k_{n,m}'^2)^{-1} (\Psi_{n,m}^{\pm}, \cdot) \Psi_{n,m}^{\pm} \quad (6.10)$$

$$JK_1 = \sum_{n,m} \pi^{-1/2} (1 + k_{n,m}'^2)^{-1} \left(1 - \frac{n^2}{k_{n,m}'^2}\right)^{-1/2} (\Psi_{n,m}^{\pm}, \cdot) e^{\pm i n \theta} \quad (6.11)$$

$$(JK_1)^* = \sum_{n,m} \pi^{-1/2} (1 + k_{n,m}'^2)^{-1} \left(1 - \frac{n^2}{k_{n,m}'^2}\right)^{-1/2} (e^{\pm i n \theta}, \cdot) \Psi_{n,m}^{\pm}, \quad (6.12)$$

yielding

$$(JK_1)^* e^{\pm i n \theta} = \sum_{m \geq 1} 2\pi^{1/2} (1 + k_{n,m}'^2)^{-1} \left(1 - \frac{n^2}{k_{n,m}'^2}\right)^{-1/2} \Psi_{n,m}^{\pm} \quad (6.13)$$

$$\|(JK_1)^* e^{\pm i n \theta}\|_{L^2(D)}^2 = \sum_{m \geq 1} 4\pi (1 + k_{n,m}'^2)^{-2} \left(1 - \frac{n^2}{k_{n,m}'^2}\right)^{-1} \quad (6.14)$$

Let us now compute the operator \check{H} .

An elementary computation yields that the solution of the boundary value problem,

$$\begin{cases} -\Delta u + u &= 0 & \text{in } D \\ u &= e^{\pm i n \theta} & \text{on } C \end{cases} \quad (6.15)$$

is given by,

$$u_n(r, \theta) = \frac{J_n(ir)}{J_n(i)} e^{\pm i n \theta} \quad (6.16)$$

Hence, the functions $e^{\pm i n \theta}$, $n \in \mathbb{N}$ belong to the domain of \check{H} , which we denote by $D(\check{H})$ and

$$\check{H} e^{\pm i n \theta} = \frac{\partial u_n}{\partial \nu} = \frac{\partial u_n(r, \theta)}{\partial r} \Big|_{r=1} = i \frac{J_n'(i)}{J_n(i)} e^{\pm i n \theta} \quad (6.17)$$

That is, the eigenvalues of \check{H} are

$$\check{\lambda}_n = i \frac{J_n'(i)}{J_n(i)} \text{ with respective associated eigenfunctions } e^{\pm i n \theta}, \quad \forall n \in \mathbb{N}. \quad (6.18)$$

Observe that each eigenvalue is a double eigenvalue except $\check{\lambda}_0$.
Set $L^2(C) := L^2(C, \frac{d\theta}{2\pi})$, then

$$\begin{aligned} D(\check{H}) &= \left\{ \varphi \in L^2(C) : \sum_{n \in \mathbb{N}} \check{\lambda}_n^2 |(\varphi, e^{in\theta})_{L^2(C)}|^2 < \infty \right\} \\ \check{H}\varphi &= \sum_{n \in \mathbb{N}} \check{\lambda}_n (\varphi, e^{in\theta})_{L^2(C)} e^{in\theta}, \quad \forall \varphi \in D(\check{H}). \end{aligned} \quad (6.19)$$

In other words, if we set $(c_n)_{n \in \mathbb{Z}}$ the Fourier coefficients of $\varphi \in L^2(C)$ and since we consider real-valued functions, then $\varphi \in D(\check{H})$ if and only if

$$\sum_{n \in \mathbb{N}} \check{\lambda}_n^2 |c_n|^2 < \infty. \quad (6.20)$$

This observation leads to a full description of $D(\check{H})$:

Proposition 6.1. *1. For each $n \in \mathbb{N}$, we have $n < \check{\lambda}_n < n + 1/2$.*

2. It follows that $\varphi \in L^2(C)$ belongs to $D(\check{H})$ if and only if

$$\sum_{n \in \mathbb{N}} n^2 |c_n|^2 < \infty.$$

Proof. The second assertion follows from the first one, which we proceed to prove.
From the recursion relations between Bessel functions and their derivatives one has

$$\check{\lambda}_n = i \frac{J'_n(i)}{J_n(i)} = n - i \frac{J_{n+1}(i)}{J_n(i)} \quad \forall n \in \mathbb{N}. \quad (6.21)$$

Since $J_n(i) = (\frac{i}{2})^n \sum_{k=0}^{\infty} \frac{1}{2^{2k} k! (n+k)!}$ $\forall n \in \mathbb{N}$ it follows that

$$n < \check{\lambda}_n < n + \frac{1}{2}, \quad \forall n \in \mathbb{N}, \quad (6.22)$$

which finishes the proof. \square

Now we turn our attention to compute explicitly the operators $\check{H}^s JK$, as they are involved in the trace-class convergence as well as in the asymptotic developments. Especially we shall prove that the limiting exponent $r = 1$ in Proposition 3.1 is excluded.
Let $s \in (0, 3/2]$. Relying on formulae (6.12)–(6.19) and owing to the fact that $\check{H}^{3/2} JK_1$ is bounded, we obtain

$$\begin{aligned} \check{H}^s JK_1 &= \sum_{n \in \mathbb{N}, m \geq 1} \frac{2\pi^{1/2} \check{\lambda}_n^s k'_{n,m}}{(1 + k'^2_{n,m})(k'^2_{n,m} - n^2)^{1/2}} (\Psi_{n,m}^{\pm}, \cdot) e^{\pm in\theta} \\ &= \sum_{n \in \mathbb{N}, m \geq 1} \check{\lambda}_n^s \theta_{n,m} (\Psi_{n,m}^{\pm}, \cdot) e^{\pm in\theta} = \sum_{n \in \mathbb{N}} \check{\lambda}_n^s (\tilde{\Psi}_n^{\pm}, \cdot) e^{\pm in\theta} \end{aligned} \quad (6.23)$$

where

$$\theta_{n,m} := \frac{2\pi^{1/2}k'_{n,m}}{(1+k'^2_{n,m})(k'^2_{n,m}-n^2)^{1/2}}, \quad \tilde{\Psi}_n^\pm := \sum_{m \geq 1} \theta_{n,m} \Psi_{n,m}^\pm, \quad (6.24)$$

Let us note that the family Ψ_n^\pm is orthogonal in $L^2(D)$. Hence setting

$$\gamma_n^2 := \|\tilde{\Psi}_n^\pm\|_{L^2(D)}^2 = \sum_{m \geq 1} \theta_{n,m}^2, \quad \phi_n^\pm := \gamma_n^{-1} \tilde{\Psi}_n^\pm, \quad (6.25)$$

we obtain that

$$\check{H}^s JK_1 = \sum_{n \in \mathbb{N}} \check{\lambda}_n^s \gamma_n(\phi_n^\pm, \cdot) e^{\pm i n \theta}, \quad (\check{H}^s JK_1)^* \check{H}^s JK_1 = \sum_{n \in \mathbb{N}} \check{\lambda}_n^{2s} \gamma_n^2(\phi_n^\pm, \cdot) \phi_n^\pm. \quad (6.26)$$

In particular we derive:

Proposition 6.2. *1. The following representation for D_∞ holds true*

$$D_\infty = (-\Delta_N + 1)^{-1} - (-\Delta_D + 1)^{-1} = \sum_{n \in \mathbb{N}} \check{\lambda}_n \gamma_n(\phi_n^\pm, \cdot) \phi_n^\pm. \quad (6.27)$$

2.

$$\lim_{\beta \rightarrow \infty} \beta \|D_\beta - D_\infty\| = \sum_{n \in \mathbb{N}} \check{\lambda}_n^2 \gamma_n^2.$$

Proof. Claim 1. is consequence of formulae (2.6)-(6.26), whereas claim 2. comes from Theorem 3.1 together with (6.23). \square

Now we proceed to prove that trace-class convergence with maximal rate, i.e. a rate proportional to $1/\beta$ does not hold true.

Theorem 6.1. *The operator $\check{H} JK_1$ is not a Hilbert–Schmidt operator. Consequently*

$$\lim_{\beta \rightarrow \infty} \beta \|D_\beta - D_\infty\|_{S_1} = \infty. \quad (6.28)$$

Proof. By [BAB12, Theorem 2.3-b], trace-class convergence with maximal rate holds true if and only if the operator $\check{H} JK_1$ is a Hilbert–Schmidt operator. Hence we are led to prove that $\|\check{H} JK_1\|_{S_2} = \infty$.

Let (f_i) be an orthonormal basis of $L^2(D)$. As $(e^{\pm i n \theta})_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(C)$, we achieve

$$\begin{aligned} \check{H} JK_1 f_j &= \sum_{n \in \mathbb{N}} (e^{\pm i n \theta}, \check{H} JK_1 f_j)_{L^2(C)} e^{\pm i n \theta} \\ &= \sum_{n \in \mathbb{N}} (\check{H} e^{\pm i n \theta}, JK_1 f_j)_{L^2(C)} e^{\pm i n \theta} \\ &= \sum_{n \in \mathbb{N}} \check{\lambda}_n (e^{\pm i n \theta}, JK_1 f_j)_{L^2(C)} e^{\pm i n \theta}. \end{aligned} \quad (6.29)$$

Yielding,

$$\|\check{H}JK_1f_j\|_{L^2(C)}^2 = \sum_{n \in \mathbb{N}} \check{\lambda}_n^2 |(e^{\pm i n \theta}, JK_1f_j)_{L^2(C)}|^2. \quad (6.30)$$

Thus

$$\begin{aligned} \|\check{H}JK_1\|_{S_2}^2 &= \sum_j \|\check{H}JK_1f_j\|_{L^2(C)}^2 = \sum_j \sum_n \check{\lambda}_n^2 |(e^{\pm i n \theta}, JK_1f_j)_{L^2(C)}|^2 \\ &= \sum_n \check{\lambda}_n^2 \sum_j |((JK_1)^* e^{\pm i n \theta}, f_j)_{L^2(D)}|^2 \\ &= \sum_n \check{\lambda}_n^2 \|(JK_1)^* e^{\pm i n \theta}\|_{L^2(D)}^2. \end{aligned} \quad (6.31)$$

Having formula (6.12) in mind we get

$$\begin{aligned} \|\check{H}JK_1\|_2^2 &= \check{\lambda}_0^2 \|(JK_1)^* 1\|_{L^2(D)}^2 + 2 \sum_{n \geq 1} \check{\lambda}_n^2 \|(JK_1)^* e^{i n \theta}\|_{L^2(D)}^2 \\ &= \check{\lambda}_0^2 \|(JK_1)^* 1\|_{L^2(D)}^2 + \sum_{n \geq 1} \frac{8\pi \check{\lambda}_n^2 k_{n,1}'^2}{(1 + k_{n,1}'^2)^2 (k_{n,1}'^2 - n^2)} \\ &\quad + \sum_{n \geq 1, m \geq 2} \frac{8\pi \check{\lambda}_n^2 k_{n,m}'^2}{(1 + k_{n,m}'^2)^2 (k_{n,m}'^2 - n^2)}. \end{aligned} \quad (6.32)$$

Now we have to investigate the behavior of $k_{n,m}'$ for large n and m .

According to [QW99], one has for, $n, m \geq 1$,

$$n + 2^{-1/3} a_m n^{1/3} < k_{n,m} < n + 2^{-1/3} a_m n^{1/3} + \frac{3}{10} a_m^2 n^{-1/3} \quad (6.33)$$

where, a_m is the m^{th} positive root of the equation,

$$A_i(-x) = \frac{1}{3} \sqrt{x} (J_{1/3}(\frac{2}{3} x^{\frac{3}{2}}) + J_{-1/3}(\frac{2}{3} x^{\frac{3}{2}})) = 0 \quad (6.34)$$

and A_i is the Airy function.

In the following, c denotes different positive constants.

For large m one has (see [AS84]), $a_m \sim c m^{2/3}$. Accordingly, there exists a positive constant c such that for $n, m \geq 1$,

$$n + c m^{2/3} n^{1/3} < k_{n,m} < n + c m^{2/3} n^{1/3} + c m^{4/3} n^{-1/3} \quad (6.35)$$

On the other hand, it is known that the zeroes of J_n and J_n' are interlaced in the following manner:

$$n \leq \dots < k_{n,m}' < k_{n,m} < k_{n,m+1}' < k_{n,m+1} < \dots \quad (6.36)$$

Hence for $n \geq 1, m \geq 2$ one has,

$$n + c(m-1)^{2/3} n^{1/3} < k_{n,m}' < n + c m^{2/3} n^{1/3} + c m^{4/3} n^{-1/3} \quad (6.37)$$

Furthermore one has (see [AS84]),

$$\begin{aligned} k'_{0,1} &= 0, \quad k'_{0,m} = k_{1,m-1} \quad \forall m \geq 2 \quad (\text{since } J'_0(z) = -J_1(z)) \\ k'_{n,1} &\sim n + cn^{1/3} \quad \text{for large } n. \end{aligned} \quad (6.38)$$

Using the latter asymptotic together with the fact that $\check{\lambda}_n \sim n$ we obtain:

$$\sum_{n \geq 1} \frac{8\pi \check{\lambda}_n^2 k'^2_{n,1}}{(1 + k'^2_{n,1})^2 (k'^2_{n,1} - n^2)} < \infty. \quad (6.39)$$

Consequently, $\|\check{H}^s JK_1\|_{S_2}$ is finite if and only if $\sum_{n \geq 1, m \geq 2} \frac{n^2}{k'^2_{n,m} (k'^2_{n,m} - n^2)}$ is finite.

However, relying on the comparison (6.37), we get

$$\begin{aligned} \sum_{m \geq 2} \frac{1}{k'^2_{n,m} (k'^2_{n,m} - n^2)} &\geq \sum_{m \geq 2} \frac{1}{(2n + cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3})^3 (cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3})} \\ &= \sum_{m \geq 2} \frac{1}{cn^{10/3}m^{2/3}(2 + cm^{2/3}n^{-2/3} + cm^{4/3}n^{-4/3})^3(1 + m^{2/3}n^{-2/3})} \\ &\geq \frac{1}{cn^{10/3}} \int_2^\infty \frac{1}{x^{2/3}(2 + cx^{2/3}n^{-2/3} + cx^{4/3}n^{-4/3})^3(1 + x^{2/3}n^{-2/3})} dx \\ &= \frac{1}{cn^3} \int_{2/n}^\infty \frac{1}{u^{2/3}(2 + cu^{2/3} + cu^{4/3})^3(1 + u^{2/3})} du \\ &\sim \frac{1}{cn^3} \int_0^\infty \frac{1}{u^{2/3}(2 + cu^{2/3} + cu^{4/3})^3(1 + u^{2/3})} du = \frac{c}{n^3} \end{aligned}$$

Therefore, $\sum_{n \geq 1, m \geq 2} \frac{n^2}{k'^2_{n,m} (k'^2_{n,m} - n^2)} = \infty$ and $\|\check{H} JK_1\|_2 = \infty$, which finishes the proof. \square

By the end of this section we shall utilize Theorem 5.1 to perform second order asymptotic for the eigenvalues of H_β . Accordingly for ϵ small enough and sufficiently large β , the Laplacian with Robin boundary conditions H_β has exactly two eigenvalues counted according to their multiplicities, for $n \geq 1$, $m \geq 1$ and only one for $n = 0$, $m \geq 1$ in the ball $B(k^2_{n,m}, \epsilon)$.

Theorem 6.2. *Set $\lambda_{n,m}^{(\beta)}$, $n \in \mathbb{N}$, $m \geq 1$ the eigenvalues of H_β . Then*

$$\lambda_{n,m}^{(\beta)} = k^2_{n,m} - \frac{2k^2_{n,m}}{\beta} + \frac{\alpha_{n,m}}{\beta^2} + o\left(\frac{1}{\beta^2}\right), \quad n \in \mathbb{N}, \quad m \geq 1 \text{ for large } \beta, \quad (6.40)$$

with,

$$\alpha_{n,m} = 2ik^2_{n,m} \frac{J'_n(i)}{J_n(i)} + \frac{4k^4_{n,m}}{1 + k^2_{n,m}} + \sum_{q \neq m} \frac{4(1 + k^2_{n,m})k^2_{n,m}k^2_{n,q}}{(1 + k^2_{n,q})(k^2_{n,m} - k^2_{n,q})}. \quad (6.41)$$

Proof. Using formulae (6.3), (6.8) and the recursion relation $J'_n(z) = \frac{n}{z}J_n(z) - J_{n+1}(z)$, we obtain that the normalized Dirichlet eigenfunctions associated to the eigenvalue $\lambda_{n,m} = k_{n,m}^2$ are given by,

$$f_1(r, \theta) = \pi^{-1/2} \frac{J_n(k_{n,m}r)}{J_{n+1}(k_{n,m})} e^{in\theta}, \quad f_2(r, \theta) = \pi^{-1/2} \frac{J_n(k_{n,m}r)}{J_{n+1}(k_{n,m})} e^{-in\theta} \quad (6.42)$$

In particular, we obtain

$$\frac{\partial f_{1,2}}{\partial r} := \frac{\partial f_{1,2}(r, \theta)}{\partial r} \Big|_{r=1} = \pi^{-1/2} k_{n,m} \frac{J'_n(k_{n,m})}{J_{n+1}(k_{n,m})} e^{\pm in\theta} = -\pi^{-1/2} k_{n,m} e^{\pm in\theta}. \quad (6.43)$$

Then, for $p, q \in \{1, 2\}$

$$\left(\frac{\partial f_p}{\partial r}, \frac{\partial f_q}{\partial r} \right)_{L^2(C)} = 2k_{n,m}^2 \delta_{p,q} \quad (6.44)$$

Moreover,

$$\mathcal{R}\left(\frac{\partial f_{1,2}}{\partial r}\right) = -\pi^{-1/2} k_{n,m} \frac{J_n(ir)}{J_n(i)} e^{\pm in\theta} \quad (6.45)$$

$$\left(\frac{\partial}{\partial r} \mathcal{R}\left(\frac{\partial f_p}{\partial r}\right), \frac{\partial f_q}{\partial r} \right)_{L^2(C)} = 2ik_{n,m}^2 \frac{J'_n(i)}{J_n(i)} \delta_{p,q}. \quad (6.46)$$

On the other hand, if $E_{n,m}$ is the eigenspace associated to the eigenvalue $k_{n,m}^2$, one has

$$E_{n,m} = \text{Vect}(f_1, f_2) \text{ and } E_{n,m}^\perp = \overline{\text{Vect}}\left(\varphi_{p,q}(r, \theta) = \pi^{-1/2} \frac{J_p(k_{p,q}r)}{J_{p+1}(k_{p,q})} e^{\pm ip\theta}, (p, q) \neq (n, m)\right)$$

Consequently,

$$a_{i,j,k} = \left(\frac{\partial f_i}{\partial r}, \frac{\partial f_k}{\partial r} \right)_{L^2(C)} \left(\frac{\partial f_j}{\partial r}, \frac{\partial f_k}{\partial r} \right)_{L^2(C)} = 4k_{n,m}^4 \delta_{i,k} \delta_{j,k}, \quad (6.47)$$

$$\frac{1}{1 + k_{n,m}^2} \sum_{f_k \in E_{n,m}} a_{i,j,k} = \frac{4k_{n,m}^4}{1 + k_{n,m}^2} \delta_{i,j}, \quad (6.48)$$

$$a_i = \left(\frac{\partial f_i}{\partial r}, \frac{\partial \varphi_{p,q}}{\partial r} \right)_{L^2(C)} = 2k_{n,m} k_{p,q} \delta_{\pm n, \pm p}, \quad (6.49)$$

and

$$\sum_{\varphi_{p,q} \in E_{n,m}} \frac{(1 + k_{n,m}^2)}{(1 + k_{p,q}^2)(k_{n,m}^2 - k_{p,q}^2)} a_i a_j = \sum_{q \neq m} \frac{4(1 + k_{n,m}^2)k_{n,m}^2 k_{n,q}^2}{(1 + k_{n,q}^2)(k_{n,m}^2 - k_{n,q}^2)} \delta_{i,j}. \quad (6.50)$$

Finally, the desired asymptotic expansion (6.40) is immediate from Theorem 5.1 and formulae (6.44), (6.46), (6.48), (6.50). \square

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